UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE	:	**MATH7202**
ASSESSMENT PATTERN	:	**MATH7202A**
MODULE NAME	:	**Algebra 4: Groups and Rings**
DATE	:	**29-Apr-08**
TIME	:	**14:30**
TIME ALLOWED	:	**2 Hours 0 Minutes**

2007/08-MATH7202A-001-EXAM-181

All questions may be attempted but only marks obtained on the best **four** *solutions will count.*
The use of an electronic calculator is **not** *permitted in this examination.*

1. Let G be a finite group. Explain what is meant by the *order*, $\operatorname{ord} g$, of $g \in G$.

   Define the *kernel* $\operatorname{Ker}(\varphi)$ and *image* $\operatorname{Im}(\varphi)$ of a group homomorphism $\varphi : G \to H$.

   State and prove a relationship which holds between $\operatorname{Ker}(\varphi)$ and $\operatorname{Im}(\varphi)$.

   Deduce that if $x \in G$ then $\operatorname{ord} \varphi(x)$ divides both $|G|$ and $|H|$.

   Let $\varphi_m : C_n \to C_n$ denote the homomorphism $\varphi_m(x^t) = x^{mt}$. State a necessary and sufficient condition on m for φ_m to be an automorphism.

   Describe $\operatorname{Aut}(C_{20})$ explicitly as a product of cyclic groups.

2. Let $\circ : G \times X \to X$ be a left action of a finite group G on a finite set X, and let $x \in X$. Explain what is meant by

   i) the orbit, $\langle x \rangle$, of $x \in X$;

   ii) the stability subgroup G_x.

   Prove that

   iii) if $y \in X$ then *either* $\langle x \rangle = \langle y \rangle$ *or* $\langle x \rangle \cap \langle y \rangle = \emptyset$, and

   iv) show there exists a bijection $\langle x \rangle \leftrightarrow G/G_x$.

   Explain what is meant by the Class Equation of such an action, and describe it explicitly in the case where $X = G = D_{14}$, the dihedral group of order 14, and the action is *conjugation* $\circ : D_{14} \times D_{14} \to D_{14}$; $g \circ h = ghg^{-1}$.

3. Let p be a prime and P a group of order p^n acting on a finite set X with fixed point set X^P. Prove that $|X^P| \equiv |X| \pmod p$.

   Let G be a group of order kp^n where k is coprime to p, and let N_p be the number of subgroups of order p^n. Under the assumption that $N_p \neq 0$, show that

   $$N_p \equiv 1 \pmod p.$$

   Suppose that G is a group of order 56 ; show that *either*

   i) G has a normal subgroup of order 7 *or*

   ii) G has a normal subgroup of order 8.

4. If $\varphi : Q \to \mathrm{Aut}(K)$ is a group homomorphism, explain what is meant by the *semi-direct product* $$K \rtimes_{\varphi} Q\,.$$

State and prove a criterion which allows one to assert that a group G is a semi-direct product of the above form where K, Q are subgroups of G.

Let p, q be primes such that $q^n < p$ and let G be a group of order pq^n. Assuming Sylow's Theorem, prove that G is a semi-direct product of the above form where $|K| = p$ and $|Q| = q^n$.

Use this result to describe all groups of order 153, stating with justification the number of distinct isomorphism types obtained.

5. Let A be a commutative integral domain which contains a field $\mathbb{F}$ as a subring and is such that $\dim_{\mathbb{F}}(A)$ is finite. Show that A is a field.

Deduce that if $p(x)$ is an irreducible polynomial over a field $\mathbb{F}$ then $\mathbb{F}[x]/(p(x))$ is a field.

If $\mathbb{F}_5$ denotes the field with five elements, show that

i) $x^2 + 2$ and $x^2 + x + 2$ are both irreducible over $\mathbb{F}_5$, and that

ii) there is an isomorphism of fields $\mathbb{F}_5[x]/(x^2+2) \cong \mathbb{F}_5[x]/(x^2+x+2)$.

6. State and prove Eisenstein's Criterion.

By making a substitution of the form $x \mapsto x + a$ show that $x^{43} + 43x + 85$ is irreducible over $\mathbb{Q}$

Give the complete factorizations of the polynomials below into monic irreducible factors over $\mathbb{Q}$, justifying your answer in each case.

i) $x^{16} - 36x^8 - 405$;

ii) $x^{10} + 1$.

END OF PAPER